Journal of Sound and Vibration (2003) **259**(5), 1209–1223
doi:10.1006/jsvi.2002.5222, available online at http://www.idealibrary.com on IDEAL®

ELASTIC WAVE SCATTERING AND DYNAMIC STRESS CONCENTRATIONS IN CYLINDRICAL SHELLS WITH A CIRCULAR CUTOUT

F. M. LI, C. HU and W. H. HUANG

Department of Aerospace Engineering & Mechanics, Harbin Institute of Technology, Harbin 150001*, People's Republic of China. E-mail: llffcc@*163*.net*

(*Received* 14 *July* 2001, *and in final form* 30 *April* 2002)

In this paper, based on the theory of elastic wave motion for open cylindrical shell, wave scattering and dynamic stress concentrations in open cylindrical shells with a hole are studied by making use of small parameter perturbation methods and boundary-integral equation techniques. The boundary-integral equations and iterative imminent series of scattered waves around the cavity of the cylindrical shell are derived. By employing this method, the approximately analytical solutions of scattered waves on the edge of cutout are gained. The computational formula for getting the dynamic stress concentration factors on the contour of cavity is developed. As an example, the numerical results of these dynamic stress concentration factors are graphically presented and discussed. The analytical methods put forward in the present work have practical significances for solving the problem of elastic wave scattering and dynamic stress concentrations in cylindrical shells with a circular cutout.

1. INTRODUCTION

Circular cylindrical shells are commonly used in many engineering applications such as aircraft fuselages, deep-diving vehicles, pressure containers, and oil lines. In practical application, the shells may be with a cutout that can produce dynamic stress concentration. Dynamic stress concentrations on the contour of cutout reduce the bearing capacity of the shell structures. The problem of dynamics for the shell is much more complex than that of the plate because of the effect of the curvature of the shell to the flexural wave equation. Consequently, the curvature of the shell influences the dynamic performance of shell in a complex manner.

Due to their importance in applications, shells with cutouts in the wall have been the subject of numerous theoretical studies and experiments for the last several decades [1–5]. Dyke [6] studied the problem of stress around a circular hole in a cylindrical shell and got the stress solutions that were based on Donnell's equations of shallow shells. Using geometrically non-linear theory, Dennis and Palazotto [7] studied the static response of a cylindrical composite panel with cutouts. Hwang and Foster [8] analyzed the axisymmetric free vibration of a thin, isotropic and shallow spherical shell with a circular cutout. Using the finite element method, Sivasubramonian *et al.* [9] studied the free vibration of isotropic curved panels with cutouts. Results were presented for cylindrical panels with square cutouts. Ram and Babu [10] investigated the bending behavior of axisymmetric laminated composite shells with a cutout using the finite element method based on a higher order shear deformation theory.

The problem of wave scattering by scatterers in all kinds of engineering structures has always been studied by large numbers of researchers since the mid-20th century [11–14]. With wave functions expansion method, Pao [15] and Pao and Mow [16] first studied the problem of the flexural wave scattering and dynamic stress concentrations in thin plates with cutouts and gave an analytical solution and numerical examples. Bogan and Hinders [17] analyzed the problem of wave scattering and dynamic stress concentrations in fiber-reinforced composites with interfacial layers, and they calculated the dynamic stress concentrations of three fiber composite materials—boron fibers in an epoxy matrix, silicon-carbide fibers in a metal matrix and tungsten fibers in a steel matrix. By making use of complex function method, Liu and Hu [18] studied the flexural wave scattering and dynamic stress concentrations in Mindlin's thick plates and presented numerical solutions. Gabrielli and Finidori [19] theoretically, numerically and experimentally studied acoustic scattering by two identical spheres by highlighting the role of the symmetries of the scatterer. Sato and Shindo [20] researched the problem of multiple scattering of elastic waves in fiber-reinforced composite and analyzed the effect of interface properties on the phase velocities, attenuations of coherent waves and the effective elastic moduli.

The presence of cutouts or discontinuities in plates and shells used in many engineering applications can produce stress concentration and reduce the strength of plates and shells. So, the stress concentration problem has been the focus of many authors concerned with many subjects [15, 21–24]. The dynamic stresses and displacements around cylindrical discontinuities due to plane harmonic shear waves were studied by Mow and Mente [22], and they considered the oblique incidence case of elastic waves. Mow and Workman [23] investigated the dynamic stresses around a fluid-filled cavity and presented the formal solution of the steady state problem of elasticity theory in the form of Rayleigh-type waves propagating on a free convex or concave cylindrical cavity. Liu *et al.* [24] proposed the complex variable method and the conformal mapping technique to solve the problem of dynamic stress concentrations.

Wave functions expansion method is the main tool for solving dynamic problems described by second order partial differential equations. However, the main difficulty for solving elastic wave problems in plates and shells lies in that one cannot usually reduce higher order dynamic equations to second order partial differential ones. Due to the limitations of present mathematical and physical methods, and the complexity of engineering structures, numerical structural analysis is often performed to investigate the dynamic problems in plates and shells. Boundary-integral equation technique is an effective method to solve these problems for plates and shells [25–28]. Using an approach called domain-boundary element method (DBEM), Dirgantara and Aliabadi [29] developed boundary-integral equations for shear deformable shallow shells by coupling integral equations for shear deformable plate and two-dimensional (2-D) plane stress elasticity. Wen *et al.* [30] developed the dual reciprocity method for transforming domain integrals to boundary integrals for shear deformable plate and shell bending formulation, and employed particular solutions for three radial basis functions. Dirgantara and Aliabadi [31] presented a new boundary element formulation for fracture mechanics analysis of shear deformable shells. They derived the hyper-singular integral equations that were employed with displacement integral equations to form DBEM formulation.

The main purpose of this paper is to investigate the problems of flexural wave scattering and dynamic stress concentrations around holes in infinite open circular cylindrical shells based on the theory of bending deflection of shallow cylindrical shell. By making use of small parameter perturbation methods and boundary element theory, a boundary-integral equation method for solving this problem is employed. The boundary-integral equations and iterative imminent series of scattered waves around the cavity in the cylindrical shell are given. With this method, one can finally get the approximately analytical solutions. Computational formula of dynamic stress concentration factors on the contour of the cutout is developed. As an example, the numerical results are graphically presented and discussed.

2. GOVERNING DYNAMIC EQUATIONS OF CYLINDRICAL SHELL

The schematic of the cylindrical shell with a circular cutout and the co-ordinate system is depicted in Figure 1. The motion equations governing the normal displacement w and stress function φ are given as follows:

$$D\nabla^2\nabla^2 w + \frac{1}{R}\frac{\partial^2\varphi}{\partial Y^2} + \rho h\frac{\partial^2 w}{\partial t^2} = q, \qquad \frac{1}{Eh}\nabla^2\nabla^2\varphi - \frac{1}{R}\frac{\partial^2 w}{\partial Y^2} = 0, \tag{1a, b}$$

where $D = Eh^3/12(1 - v^2)$ is the bending stiffness of the shell wall, E Young's modulus, v the Poisson ratio, ρ the mass density, R the radius of curvature, h the thickness of the shell, X and Y the rectangular co-ordinates with Y axis oriented along the shell axis, $\nabla^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2$ the 2-D Laplacian operator in variables X and Y, t the time and q the normal load, for free vibration, $q = 0$.

The solutions of the fourth order partial differential equations (1a) and (1b) can be expressed as

$$w = \nabla^2\nabla^2 f, \qquad \varphi = \frac{Eh}{R}\frac{\partial^2 f}{\partial Y^2}. \tag{2a, b}$$

Substituting equations (2a) and (2b) into equations (1a) and (1b) gives the following form:

$$D\nabla^2\nabla^2\nabla^2\nabla^2 f + \rho h\frac{\partial^2}{\partial t^2}[\nabla^2\nabla^2 f] + \frac{Eh}{R^2}\frac{\partial^4 f}{\partial Y^4} = 0. \tag{3}$$

In order to obtain the normal displacement w and the stress function φ, one assumes the steady wave solution of equation (3) as

$$f = F(X, Y)\mathrm{e}^{-i\omega t}, \tag{4}$$

where ω and F are, respectively, the circular frequency and the complex amplitude value of bending wave. Introducing non-dimensional co-ordinates

$$x = X/a, \qquad y = Y/a, \qquad r = \sqrt{X^2 + Y^2}/a, \tag{5}$$

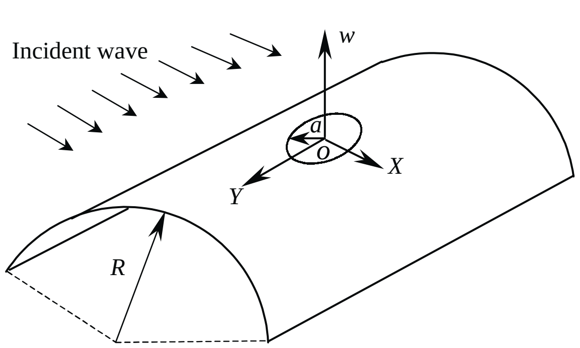

Figure 1. Schematic of cylindrical shell with a circular cutout and co-ordinate system.

where a is the characteristic length with regard to cutout in the shell, such as the radius of the cutout, and substituting equations (4) and (5) into equation (3), one can obtain

$$\nabla^2\nabla^2\nabla^2\nabla^2 F - \alpha^4\nabla^2\nabla^2 F + \varepsilon\partial^4 F/\partial y^4 = 0, \tag{6}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2-D Laplacian operator in variables x and y, $\varepsilon = 12(1-v^2)(a/\sqrt{Rh})^4$ the small structural parameter and $\alpha = ka = 2\pi a/\lambda$ the non-dimensional wavenumber, in which $k = [\rho h\omega^2/D]^{1/4}$ is the wavenumber and λ the wavelength.

The solution of equation (6) is assumed to have the form

$$F = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots = \sum_{m=0}^{+\infty} \varepsilon^m F_m. \tag{7}$$

Substituting equation (7) into equation (6) and letting the coefficients of different powers of ε on the left side of equation (6) be equal to zero give the following expression:

$$\nabla^2\nabla^2[\nabla^2\nabla^2 - \alpha^4]F_m = -\mathrm{H}(m-1)\partial^4 F_{m-1}/\partial y^4, \tag{8}$$

where $\mathrm{H}(\cdot)$ denotes the Heaviside function.

3. GENERALIZED INTERNAL FORCES AND BOUNDARY CONDITIONS

In the polar co-ordinates (r, θ), the generalized internal forces can be expressed in terms of normal displacement w and stress function φ as follows:

$$\begin{gathered} N_r = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}, \qquad N_\theta = \frac{\partial^2\varphi}{\partial r^2}, \qquad N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) \\ M_r = -D\left[v\nabla^2 w + (1-v)\frac{\partial^2 w}{\partial r^2}\right], \quad M_\theta = -D\left[\nabla^2 w - (1-v)\frac{\partial^2 w}{\partial r^2}\right] \\ M_{r\theta} = -D(1-v)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right), \\ Q_r = -D\frac{\partial}{\partial r}(\nabla^2 w), \qquad Q_\theta = -D\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2 w) \end{gathered} \tag{9}$$

where N_r, N_θ and $N_{r\theta}$ are the internal membrane forces, M_r, M_θ and $M_{r\theta}$ the bending moments, Q_r and Q_θ the transverse shear forces, respectively, $\nabla^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial^2/\partial\theta^2$ the 2-D Laplacian operator in variables r and θ. The nominal shear force is defined as

$$V_r = Q_r + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta}. \tag{10}$$

In the polar co-ordinates (r, θ), it can be written as

$$V_r = -D\frac{\partial}{\partial r}(\nabla^2 w) - D(1-v)\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2}\right). \tag{11}$$

For circular boundary, the generalized internal forces on the boundary can be expressed as

$$(\cdot)_n = (\cdot)_r, \qquad (\cdot)_t = (\cdot)_\theta, \qquad (\cdot)_{nt} = (\cdot)_{r\theta}, \tag{12}$$

where $(\cdot)$ represents, respectively, the internal membrane force, the bending moment and the nominal shear force, and the subscripts n and t are the normal and tangent directions. In the present study, the boundary of the circular hole is assumed to be free of traction. So, the boundary conditions can be written as

$$(M_r)_{r=1} = 0, \qquad (V_r)_{r=1} = 0. \tag{13}$$

4. ASYMPTOTIC SOLUTIONS

When the subscript m in equation (8) is equal to zero, the following homogenous equation can be gained:

$$\nabla^2\nabla^2[\nabla^2\nabla^2 - \alpha^4]F_0 = 0. \tag{14}$$

The solution of equation (14) can be written as follows:

$$F_0 = F_0^{(1,2)} + F_0^{(3,4)}, \tag{15}$$

where $F_0^{(1,2)}$ and $F_0^{(3,4)}$ satisfy the following equations:

$$\nabla^2\nabla^2 F_0^{(1,2)} - \alpha^4 F_0^{(1,2)} = 0, \qquad \nabla^2\nabla^2 F_0^{(3,4)} = 0. \tag{16a, b}$$

The solution of equation (16a) is given as

$$F_0^{(1,2)} = F_0^{(1)} + F_0^{(2)}, \tag{17}$$

where $F_0^{(1)}$ and $F_0^{(2)}$ satisfy the following Helmholtz and modified Helmholtz equations:

$$(\nabla^2 + \alpha^2)F_0^{(1)} = 0, \qquad (\nabla^2 - \alpha^2)F_0^{(2)} = 0. \tag{18a, b}$$

In the polar co-ordinates (r, θ), the solutions of equations (18a) and (18b) can be written as [32]

$$F_0^{(1,2)} = F_0^{(1)} + F_0^{(2)} = \sum_{n=-\infty}^{+\infty} \left[A_n^0 H_n^{(1)}(\alpha r)e^{in\theta} + B_n^0 K_n(\alpha r)e^{in\theta}\right], \tag{19}$$

where A_n^0 and B_n^0 are unknown coefficients which are determined by the boundary conditions of the cavity, $H_n^{(1)}(\cdot)$ the nth order Hankel function of the first kind and $K_n(\cdot)$ the nth order modified Bessel function.

The fundamental solution of equation (16b) is the one corresponding to the following equation:

$$\nabla^2\nabla^2 F_0^{*(3,4)} = \delta(P, Q) = \delta(\mathbf{r}_P - \mathbf{r}_Q), \tag{20}$$

where $\delta(\mathbf{r}_P - \mathbf{r}_Q)$ is the Dirac delta function, $\mathbf{r}_Q$ the vector representing a unit applied potential at a given point Q (source point) and $\boldsymbol{r}_P$ the variable corresponding to the observation point P. The fundamental solution for equations such as equation (20) is a function only of the distance between the source point and the observation point. This distance is denoted by ξ as seen in Figure 2 and defined as

$$\xi = |\boldsymbol{\xi}| = |\mathbf{r}_P - \mathbf{r}_Q|. \tag{21}$$

The fundamental solution of equation (20) can be given by [33, 34]

$$F_0^{*(3,4)}(P, Q) = \frac{\xi^2}{8\pi}\ln\xi. \tag{22}$$

The integral equations for solving $F_0^{(3,4)}$ can be derived by the method of integration by parts. Integrating the integral $\int_\Gamma(\nabla^2\nabla^2 F_0^{(3,4)})F_0^{*(3,4)}\,\mathrm{d}\Omega$ by parts twice, the following

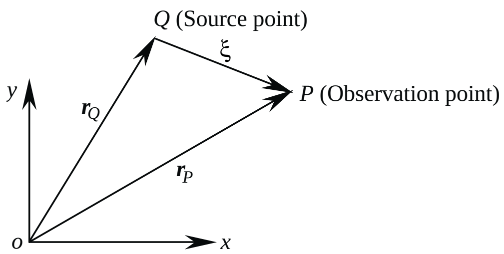

Figure 2. Notation for source and observation points.

expression can be obtained:

$$F_0^{(3,4)}(Q) = \int_\Gamma F_0^{(3,4)}(P_0)\frac{\partial[\nabla^2 F_0^{*\,(3,4)}(P_0,Q)]}{\partial n(P_0)}\,\mathrm{d}\Gamma - \int_\Gamma \frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\nabla^2 F_0^{*\,(3,4)}(P_0,Q)\,\mathrm{d}\Gamma + \int_\Gamma \frac{\partial F_0^{*\,(3,4)}(P_0,Q)}{\partial n(P_0)}\nabla^2 F_0^{(3,4)}(P_0)\,\mathrm{d}\Gamma - \int_\Gamma \nabla^2\left[\frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\right]F_0^{*\,(3,4)}(P_0,Q)\,\mathrm{d}\Gamma, \tag{23}$$

where $P_0 \in \Gamma$ and $Q \in \Omega$ are the field points on the boundary and inside the domain respectively, $n(P_0)$ the normal direction on the boundary. Integrating the last two terms at the right side of equation (23) by parts twice, the following equations can be gained:

$$\int_\Gamma \frac{\partial F_0^{*\,(3,4)}(P_0,Q)}{\partial n(P_0)}\nabla^2 F_0^{(3,4)}(P_0)\,\mathrm{d}\Gamma = \int_\Gamma \nabla^2\left[\frac{\partial F_0^{*\,(3,4)}(P_0,Q)}{\partial n(P_0)}\right]F_0^{(3,4)}(P_0)\,\mathrm{d}\Gamma + \frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\frac{\partial F_0^{*\,(3,4)}(P_0,Q)}{\partial n(P_0)}\bigg|_\Gamma - \frac{\partial^2 F_0^{*\,(3,4)}(P_0,Q)}{\partial n^2(P_0)}F_0^{(3,4)}(P_0)\bigg|_\Gamma, \tag{24a}$$

$$\int_\Gamma \nabla^2\left[\frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\right]F_0^{*\,(3,4)}(P_0,Q)\,\mathrm{d}\Gamma = \int_\Gamma \nabla^2 F_0^{*\,(3,4)}(P_0,Q)\frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\,\mathrm{d}\Gamma + \frac{\partial^2 F_0^{(3,4)}(P_0)}{\partial n^2(P_0)}F_0^{*\,(3,4)}(P_0,Q)\bigg|_\Gamma - \frac{\partial F_0^{*\,(3,4)}(P_0,Q)}{\partial n(P_0)}\frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\bigg|_\Gamma. \tag{24b}$$

Substituting equations (24a) and (24b) into equation (23), one can get

$$F_0^{(3,4)}(Q) = 2\int_\Gamma F_0^{(3,4)}(P_0)\frac{\partial[\nabla^2 F_0^{*\,(3,4)}(P_0,Q)]}{\partial n(P_0)}\,\mathrm{d}\Gamma - 2\int_\Gamma \frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\nabla^2 F_0^{*\,(3,4)}(P_0,Q)\,\mathrm{d}\Gamma + 2\frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\frac{\partial F_0^{*\,(3,4)}(P_0,Q)}{\partial n(P_0)}\bigg|_\Gamma - \frac{\partial^2 F_0^{*\,(3,4)}(P_0,Q)}{\partial n^2(P_0)}F_0^{(3,4)}(P_0)\bigg|_\Gamma - \frac{\partial^2 F_0^{(3,4)}(P_0)}{\partial n^2(P_0)}F_0^{*\,(3,4)}(P_0,Q)\bigg|_\Gamma. \tag{25}$$

By taking the point Q to the boundary, that is $Q \in \Omega \to Q_0 \in \Gamma$, equation (25) can be written as follows:

$$\begin{aligned} C(Q)F_0^{(3,4)}(Q) = &2\int_\Gamma F_0^{(3,4)}(P_0)\frac{\partial[\nabla^2 F_0^{*(3,4)}(P_0,Q)]}{\partial n(P_0)}\,\mathrm{d}\Gamma - 2\int_\Gamma \frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\nabla^2 F_0^{*(3,4)}(P_0,Q)\,\mathrm{d}\Gamma \\ &+ 2\frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\frac{\partial F_0^{*(3,4)}(P_0,Q)}{\partial n(P_0)}\bigg|_\Gamma - \frac{\partial^2 F_0^{*(3,4)}(P_0,Q)}{\partial n^2(P_0)}F_0^{(3,4)}(P_0)\bigg|_\Gamma \\ &- \frac{\partial^2 F_0^{(3,4)}(P_0)}{\partial n^2(P_0)}F_0^{*(3,4)}(P_0,Q)\bigg|_\Gamma. \end{aligned} \tag{26a}$$

In equation (26a) there are altogether three unknown boundary variables, namely, $F_0^{(3,4)}(P_0)$, $\partial F_0^{(3,4)}(P_0)/\partial n(P_0)$ and $\partial^2 F_0^{(3,4)}(P_0)/\partial n^2(P_0)$. So the following two equations are required to determine $F_0^{(3,4)}(Q)$:

$$\begin{aligned} C(Q)\frac{\partial F_0^{(3,4)}(Q)}{\partial n(Q)} = &2\int_\Gamma F_0^{(3,4)}(P_0)\frac{\partial^2[\nabla^2 F_0^{*(3,4)}(P_0,Q)]}{\partial n(P_0)\partial n(Q)}\mathrm{d}\Gamma \\ &- 2\int_\Gamma \frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\frac{\partial[\nabla^2 F_0^{*(3,4)}(P_0,Q)]}{\partial n(Q)}\mathrm{d}\Gamma \\ &+ 2\frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\frac{\partial^2 F_0^{*(3,4)}(P_0,Q)}{\partial n(P_0)\partial n(Q)}\bigg|_\Gamma \\ &- \frac{\partial^3 F_0^{*(3,4)}(P_0,Q)}{\partial n^2(P_0)\partial n(Q)}F_0^{(3,4)}(P_0)\bigg|_\Gamma - \frac{\partial^2 F_0^{(3,4)}(P_0)}{\partial n^2(P_0)}\frac{\partial F_0^{*(3,4)}(P_0,Q)}{\partial n(Q)}\bigg|_\Gamma, \end{aligned} \tag{26b}$$

$$\begin{aligned} C(Q)\frac{\partial^2 F_0^{(3,4)}(Q)}{\partial n^2(Q)} = &2\int_\Gamma F_0^{(3,4)}(P_0)\frac{\partial^3[\nabla^2 F_0^{*(3,4)}(P_0,Q)]}{\partial n(P_0)\partial n^2(Q)}\,\mathrm{d}\Gamma \\ &- 2\int_\Gamma \frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\frac{\partial^2[\nabla^2 F_0^{*(3,4)}(P_0,Q)]}{\partial n^2(Q)}\,\mathrm{d}\Gamma \\ &+ 2\frac{\partial F_0^{(3,4)}(P_0)}{\partial n(P_0)}\frac{\partial^3 F_0^{*(3,4)}(P_0,Q)}{\partial n(P_0)\partial n^2(Q)}\bigg|_\Gamma \\ &- \frac{\partial^4 F_0^{*(3,4)}(P_0,Q)}{\partial n^2(P_0)\partial n^2(Q)}F_0^{(3,4)}(P_0)\bigg|_\Gamma - \frac{\partial^2 F_0^{(3,4)}(P_0)}{\partial n^2(P_0)}\frac{\partial^2 F_0^{*(3,4)}(P_0,Q)}{\partial n^2(Q)}\bigg|_\Gamma \end{aligned} \tag{26c}$$

where $C\,(Q)$ is the jump term. The value of $C\,(Q)$ is

$$C(Q) = \begin{cases} 1/2 & \text{for } Q = Q_0 \in \Gamma, \\ 1 & \text{for } Q \in \Omega. \end{cases} \tag{27}$$

Then, the zeroth order or the unperturbed terms for determining the normal displacement and the stress function are written as

$$W_0 = \nabla^2\nabla^2 F_0, \qquad \Phi_0 = \frac{Eh}{R}\frac{\partial^2 F_0}{\partial y^2}. \tag{28}$$

$W_0^{(1)}\mathrm{e}^{-\mathrm{i}\omega t}$ and $W_0^{(2)}\mathrm{e}^{-\mathrm{i}\omega t}$ denote, respectively, the scattered and the evanescent waves of normal displacement around the hole. They compose together the scattered waves of normal displacement. $\Phi_0^{(1)}\mathrm{e}^{-\mathrm{i}\omega t}$ expresses the scattered wave of membrane force, and

$\Phi_0^{(2)}e^{-i\omega t}$ and $\Phi_0^{(3,4)}e^{-i\omega t}$ represent the evanescent waves of membrane force around the cutout. They form altogether the scattered waves of membrane force.

The first order and the above perturbation solutions can be gained by the boundary-integral equation techniques. By introducing

$$W_m = \nabla^2\nabla^2 F_m, \quad m = 1, 2, 3, \ldots, \tag{29}$$

and substituting it into equation (8), one can obtain the first order and the above perturbation equations

$$\nabla^2\nabla^2 W_m - \alpha^4 W_m = -\partial^4 F_{m-1}/\partial y^4, \quad m = 1, 2, 3, \ldots, \tag{30}$$

The fundamental solution of equation (30) should satisfy the following expression:

$$\nabla^2\nabla^2 W_m^* - \alpha^4 W_m^* = \delta(P, Q) = \delta(\mathbf{r}_P, \mathbf{r}_Q). \tag{31}$$

In the polar co-ordinates, the fundamental solution of equation (31) can be expressed as [35]

$$W_m^*(P, Q) = \frac{i}{8\alpha^2}[H_0^{(1)}(\alpha\xi) + \frac{2i}{\pi}K_0(\alpha\xi)], \tag{32}$$

where the significance of ξ is the same as that in equation (21), $H_0^{(1)}(\cdot)$ the zeroth order Hankel function of the first kind and $K_0(\cdot)$ the zeroth order modified Bessel function.

By making use of the fundamental solution, the integral formulations of the first order and the above perturbation solutions can be written as [34]

$$\begin{aligned} C(Q)DW_m(Q) = &-D\int_\Omega W_m^*(P, Q)\frac{\partial^4 F_{m-1}(P)}{\partial y^4}\,d\Omega \\ &+ \int_\Gamma \left[M_n^*(P_0, Q)\frac{\partial W_m(P_0)}{\partial n(P_0)} - M_n^{(m)}(P_0)\frac{\partial W_m^*(P_0, Q)}{\partial n(P_0)}\right] d\Gamma \\ &- \int_\Gamma [V_n^*(P_0, Q)W_m(P_0) - V_n^{(m)}(P_0)W_m^*(P_0, Q)]\,d\Gamma, \end{aligned} \tag{33a}$$

where $C(Q)$ is the jump term and defined by equation (27), D the bending stiffness of shell wall as defined in equation (1), $P_0 \in \Gamma$ and $P \in \Omega$ the observation points on the boundary and inside the domain, respectively, and the definition of Q the same as that in equation (27). $M_n^*(P_0, Q)$ and $V_n^*(P_0, Q)$ can be expressed as

$$\begin{aligned} M_n^*(P_0, Q) = M_n[W_m^*(P_0, Q)] = -D\Bigg\{&\frac{i\nu}{8\alpha r}\left[\frac{1}{2}\Big(H_{-1}^{(1)}(\alpha\xi) - H_1^{(1)}(\alpha\xi)\Big) - \frac{i}{\pi}(K_{-1}(\alpha\xi) + K_1(\alpha\xi))\right] \\ &+\frac{i}{16}\left[\frac{1}{2}\Big(H_{-2}^{(1)}(\alpha\xi) - 2H_0^{(1)}(\alpha\xi) + H_2^{(1)}(\alpha\xi)\Big) + \frac{i}{2\pi}(K_{-2}(\alpha\xi) + 2K_0(\alpha\xi) + K_2(\alpha\xi))\right]\Bigg\}, \end{aligned}$$

$$\begin{aligned} V_n^*(P_0, Q) = V_n[W_m^*(P_0, Q)] = -D\Bigg\{&\frac{i\alpha}{32}\left[\frac{1}{2}\Big(H_{-3}^{(1)}(\alpha\xi) - 3H_{-1}^{(1)}(\alpha\xi) + 3H_1^{(1)}(\alpha\xi) - H_3^{(1)}(\alpha\xi)\Big)\right. \\ &\left.-\frac{i}{\pi}(K_{-3}(\alpha\xi) + 3K_{-1}(\alpha\xi) + 3K_1(\alpha\xi) + K_3(\alpha\xi))\right] \\ &- \frac{i}{8\alpha r^2}\left[\frac{1}{2}\Big(H_{-1}^{(1)}(\alpha\xi) - H_1^{(1)}(\alpha\xi)\Big) - \frac{i}{\pi}(K_{-1}(\alpha\xi) + K_1(\alpha\xi))\right] \\ &+ \frac{i}{16r}\left[\frac{1}{2}\Big(H_{-2}^{(1)}(\alpha\xi) - 2H_0^{(1)}(\alpha\xi) + H_2^{(1)}(\alpha\xi)\Big)\right. \\ &\left.+ \frac{i}{2\pi}(K_{-2}(\alpha\xi) + 2K_0(\alpha\xi) + K_2(\alpha\xi))\right]\Bigg\}. \end{aligned}$$

In equation (33a), only two boundary variables are known in the four ones, namely, $W_m(P_0)$, $\partial W_m(P_0)/\partial n(P_0)$, $M_n^{(m)}(P_0) = M_n[W_m(P_0)]$ and $V_n^{(m)}(P_0) = V_n[W_m(P_0)]$. The other supplementary equation is necessary to obtain $W_m(Q)$

$$C(Q)D\frac{\partial W_m(Q)}{\partial n(Q)} = -\int_\Omega \frac{\partial W^*(P,Q)}{\partial n(Q)}\frac{\partial^4 F_{m-1}(P)}{\partial y^4}\,\mathrm{d}\Omega + \int_\Gamma \left[\frac{\partial M_n^*(P_0,Q)}{\partial n(Q)}\frac{\partial W_m(P_0)}{\partial n(P_0)} - M_n^{(m)}(P_0)\frac{\partial^2 W^*(P_0,Q)}{\partial n(P_0)\partial n(Q)}\right]\mathrm{d}\Gamma - \int_\Gamma \left[\frac{\partial V_n^*(P_0,Q)}{\partial n(Q)}W_m(P_0) - V_n^{(m)}(P_0)\frac{\partial W^*(P_0,Q)}{\partial n(Q)}\right]\mathrm{d}\Gamma. \quad (33b)$$

In order to determine F_m, one can rearrange equation (29) as follows:

$$\nabla^2\nabla^2 F_m = W_m, \quad m = 1,2,3,\ldots. \quad (34)$$

The fundamental solution of equation (34) should satisfy the following formulation:

$$\nabla^2\nabla^2 F_m^* = \delta(P,Q) = \delta(\mathbf{r}_P - \mathbf{r}_Q). \quad (35)$$

The fundamental solution of equation (35) is the same as equation (22), namely

$$F_m^*(P,Q) = \frac{\xi^2}{8\pi}\ln\xi, \quad (36)$$

where the significance of ξ is the same as that in equation (21). The integral equations for solving F_m can be derived by the same method for determining $F_0^{(3,4)}$ as stated above, and they are given by

$$C(Q)F_m(Q) = \int_\Omega W_m(P)F_m^*(P,Q)\,\mathrm{d}\Omega + 2\int_\Gamma F_m(P_0)\frac{\partial[\nabla^2 F_m^*(P_0,Q)]}{\partial n(P_0)}\mathrm{d}\Gamma - 2\int_\Gamma \frac{\partial F_m(P_0)}{\partial n(P_0)}\nabla^2 F_m^*(P_0,Q)\,\mathrm{d}\Gamma + 2\left.\frac{\partial F_m(P_0)}{\partial n(P_0)}\frac{\partial F_m^*(P_0,Q)}{\partial n(P_0)}\right|_\Gamma - \left.\frac{\partial^2 F_m^*(P_0,Q)}{\partial n^2(P_0)}F_m(P_0)\right|_\Gamma - \left.\frac{\partial^2 F_m(P_0)}{\partial n^2(P_0)}F_m^*(P_0,Q)\right|_\Gamma, \quad (37a)$$

$$C(Q)\frac{\partial F_m(Q)}{\partial n(Q)} = \int_\Omega W_m(P)\frac{\partial F_m^*(P,Q)}{\partial n(Q)}\mathrm{d}\Omega + 2\int_\Gamma F_m(P_0)\frac{\partial^2[\nabla^2 F_m^*(P_0,Q)]}{\partial n(P_0)\partial n(Q)}\mathrm{d}\Gamma - 2\int_\Gamma \frac{\partial F_m(P_0)}{\partial n(P_0)}\frac{\partial[\nabla^2 F_m^*(P_0,Q)]}{\partial n(Q)}\mathrm{d}\Gamma + 2\left.\frac{\partial F_m(P_0)}{\partial n(P_0)}\frac{\partial^2 F_m^*(P_0,Q)}{\partial n(P_0)\partial n(Q)}\right|_\Gamma - \left.\frac{\partial^3 F_m^*(P_0,Q)}{\partial n^2(P_0)\partial n(Q)}F_m(P_0)\right|_\Gamma - \left.\frac{\partial^2 F_m(P_0)}{\partial n^2(P_0)}\frac{\partial F_m^*(P_0,Q)}{\partial n(Q)}\right|_\Gamma, \quad (37b)$$

$$C(Q)\frac{\partial^2 F_m(Q)}{\partial n^2(Q)} = \int_\Omega W_m(P)\frac{\partial^2 F_m^*(P,Q)}{\partial n^2(Q)} + 2\int_\Gamma F_m(P_0)\frac{\partial^3[\nabla^2 F_m^*(P_0,Q)]}{\partial n(P_0)\partial n^2(Q)}\mathrm{d}\Gamma - 2\int_\Gamma \frac{\partial F_m(P_0)}{\partial n(P_0)}\frac{\partial^2[\nabla^2 F_m^*(P_0,Q)]}{\partial n^2(Q)}\mathrm{d}\Gamma + 2\left.\frac{\partial F_m(P_0)}{\partial n(P_0)}\frac{\partial^3 F_m^*(P_0,Q)}{\partial n(P_0)\partial n^2(Q)}\right|_\Gamma - \left.\frac{\partial^4 F_m^*(P_0,Q)}{\partial n^2(P_0)\partial n^2(Q)}F_m(P_0)\right|_\Gamma - \left.\frac{\partial^2 F_m(P_0)}{\partial n^2(P_0)}\frac{\partial^2 F_m^*(P_0,Q)}{\partial n^2(Q)}\right|_\Gamma. \quad (37c)$$

Then, the first order and the above perturbation terms for determining the stress function are given by

$$\Phi_m = \frac{Eh\partial^2 F_m}{R\ \partial y^2}, \quad m = 1, 2, 3, \ldots, \tag{38}$$

5. SCATTERING OF ELASTIC WAVES

Assuming that a steady flexural wave propagates in the positive x direction, one can give the incident wave of the generalized potential function F as follows:

$$\tilde{F}^{(i)} = \tilde{F}_0 \exp[\mathrm{i}(\alpha x - \omega t)], \tag{39}$$

where $\tilde{F}_0$ is the amplitude of the incident wave, α the non-dimensional wavenumber and ω the circular frequency of the incident wave. The incident wave fields of the normal displacement and the stress function on the boundary of the cutout are of the forms

$$\tilde{W}^{(i)} = \nabla^2\nabla^2\tilde{F}^{(i)} = \alpha^4\tilde{F}_0 \exp[\mathrm{i}(\alpha x - \omega t)] = \tilde{W}_0 \exp[\mathrm{i}(\alpha x - \omega t)], \tag{40a}$$

$$\tilde{\Phi}^{(i)} = \frac{Eh}{R}\frac{\partial^2 \tilde{F}^{(i)}}{\partial y^2} = 0, \tag{40b}$$

where $\tilde{W}_0 = \alpha^4 \tilde{F}_0$ denotes the amplitude of the incident wave of the normal displacement.

The scattered wave of the generalized potential function F around the cavity can be written in the form of equation (7), namely

$$\tilde{F}^{(s)} = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \ldots. \tag{41}$$

The scattered wave fields of the normal displacement and the stress function on the edge of the cutout are given by

$$\tilde{W}^{(s)} = \nabla^2\nabla^2\tilde{F}^{(s)} = \sum_{m=0}^{+\infty} \varepsilon^m \nabla^2\nabla^2 F_m = \sum_{m=0}^{+\infty} \varepsilon^m W_m, \tag{42a}$$

$$\tilde{\Phi}^{(s)} = \frac{Eh}{R}\frac{\partial^2 \tilde{F}^{(s)}}{\partial y^2} = \frac{Eh}{R}\sum_{m=0}^{+\infty} \varepsilon^m \frac{\partial^2 F_m}{\partial y^2} = \sum_{m=0}^{+\infty} \varepsilon^m \Phi_m. \tag{42b}$$

The total elastic wave field of the generalized potential function F around the cavity should be the sum of the incident wave and the scattered wave, namely

$$\tilde{F} = \tilde{F}^{(i)} + \tilde{F}^{(s)}. \tag{43}$$

Accordingly, the total elastic wave fields of the normal displacement and the stress function around the cutout are written as

$$\tilde{W} = \tilde{W}^{(i)} + \tilde{W}^{(s)}, \qquad \tilde{\Phi} = \tilde{\Phi}^{(i)} + \tilde{\Phi}^{(s)}. \tag{44a, b}$$

6. DYNAMIC STRESS CONCENTRATIONS

To determine the dynamic stress concentration factors on the contour of the cavity is the main purpose of the present investigation. Suppose that the boundary conditions on the boundary of the cutout are free of tractions. So only the hoop dynamic moment and shear force exist along the boundary curve. The dynamic stress concentration factor is defined as

$$DSCF = \left|\frac{\sigma_{N\theta}}{\sigma_{M0}}\right| + \left|\frac{\sigma_{M\theta}}{\sigma_{M0}}\right|, \tag{45}$$

where $\sigma_{N\theta}$ and $\sigma_{M\theta}$ are the stresses caused, respectively, by the hoop membrane force $\tilde{N}_\theta$ and the hoop bending moment $\tilde{M}_\theta$, and σ_{M0} is the stress amplitude value caused by the incident elastic wave. $\sigma_{N\theta}$, $\sigma_{M\theta}$ and σ_{M0} can be, respectively, given by

$$\sigma_{N\theta} = \frac{\tilde{N}_\theta}{h}, \qquad \sigma_{M\theta} = \frac{6\tilde{M}_\theta}{h^2}, \qquad \sigma_{M0} = \frac{6\tilde{M}_0}{h^2}, \tag{46a−c}$$

where h is the thickness of the shell, and $\tilde{M}_0 = -D\alpha^2 \tilde{W}_0$ is the amplitude of the bending moment caused by the incident elastic wave. Substituting equations (46a)–(46c) into equation (45), one can get

$$DSCF = \left|\frac{\tilde{M}_\theta}{\tilde{M}_0}\right|\left(1 + \frac{h}{6}\left|\frac{\tilde{N}_\theta}{\tilde{M}_\theta}\right|\right). \tag{47}$$

For the free boundary conditions of the circular cutout, the hoop membrane force $\tilde{N}_\theta$ and the hoop bending moment $\tilde{M}_\theta$ are expressed as

$$\tilde{N}_\theta = \tilde{N}_\theta + \tilde{N}_r = \nabla^2 \tilde{\Phi} = \frac{Eh}{R}\nabla^2\left(\frac{\partial^2 \tilde{F}}{\partial y^2}\right), \tag{48}$$

$$\tilde{M}_\theta = \tilde{M}_\theta + \tilde{M}_r = -D(1+v)\nabla^2 \tilde{W}, \tag{49}$$

where $\tilde{N}_r$ and $\tilde{M}_r$ are the radial membrane force and bending moment. For the case of free boundary, $\tilde{N}_r$ and $\tilde{M}_r$ are all equal to zero. By substituting equations (48) and (49) into equation (47), the following formulation for calculating the dynamic stress concentration factors can be derived:

$$\begin{aligned} DSCF &= \left|\frac{\tilde{M}_\theta}{\tilde{M}_0}\right|\left(1 + \sqrt{\frac{\varepsilon(1-v)}{3(1+v)}}\left|\frac{1}{\nabla^2 \tilde{W}}\frac{\partial^2(\nabla^2 \tilde{F})}{\partial y^2}\right|\right) \\ &= \left|\frac{(1+v)\nabla^2 \tilde{W}}{\alpha^2 \tilde{W}_0}\right|\left(1 + \sqrt{\frac{\varepsilon(1-v)}{3(1+v)}}\left|\frac{1}{\nabla^2 \tilde{W}}\frac{\partial^2(\nabla^2 \tilde{F})}{\partial y^2}\right|\right), \end{aligned} \tag{50}$$

where v is the Poisson ratio of the shell, and ε the small perturbation parameter as defined in equation (6).

7. EXAMPLE AND DISCUSSION

A steady flexural wave propagating in the positive x direction is studied. Assume that the boundary curve of the cavity is free of traction. In practical engineering, this kind of boundary condition is more common and more investigated. For different small perturbation parameters, Figure 3 shows the dynamic stress concentration factors as a function of non–dimensional wavenumber of the incident flexural wave around a circular cutout. For three cases of non-dimensional wavenumber, Figure 4 depicts the dynamic stress concentration factors varying with the structural parameter $a/\sqrt{Rh}$.

For cylindrical shells with a cutout, it can be seen in Figure 3 that the dynamic stress concentration factors decrease as the non–dimensional wavenumber α increases for the case of α being between 0·1 and 1·0. But the dynamic stress concentration factors sharply decrease as α increases when it is less than 0·25 for the cases of ε being equal to 0·2 and 0·5. For different small perturbation parameters, it shows that the dynamic stress

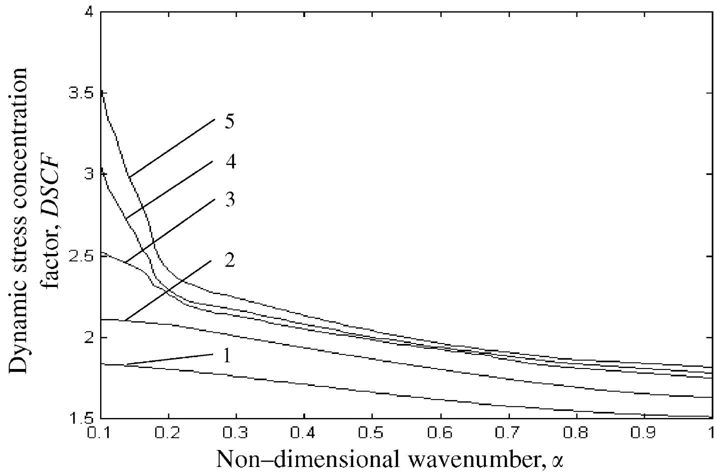

Figure 3. Dynamic stress concentration factors as a function of non-dimensional wavenumber α for open cylindrical shells with a circular cutout, $\theta = \pi/2, \nu = 0{\cdot}3$. For the five curves, No. 1 denotes thin plates, No. 2 the case of $\varepsilon = 0$ for cylindrical shells, No. 3 the case of $\varepsilon = 0{\cdot}1$ for cylindrical shells, No. 4, $\varepsilon = 0{\cdot}2$ and No. 5, $\varepsilon = 0{\cdot}5$.

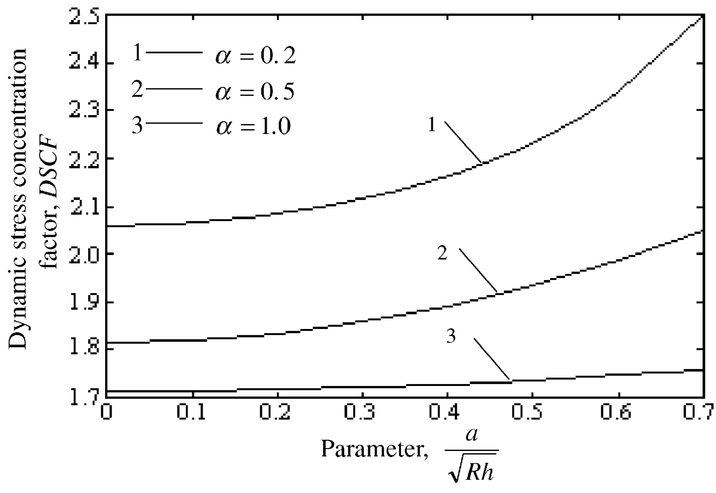

Figure 4. Dynamic stress concentration factors versus parameter $a/\sqrt{Rh}$ ($\theta = \pi/2$).

concentration factors of cylindrical shells with a circular cavity are bigger than those of thin plates with a circular hole in the whole scope of α. In general, the dynamic stress concentration factor increases as the small perturbation parameter ε increases for every non–dimensional wavenumber α. Especially due to the effect of the membrane force, the dynamic stress concentration factor is larger than that of thin plate for the case of the small perturbation parameter ε being equal to zero.

It can be seen from Figure 4 that the dynamic stress concentration factors increase as the structural parameter $a/\sqrt{Rh}$ increases for three cases of non-dimensional wavenumber. However, the dynamic stress concentration factor increases more quickly for smaller non-dimensional wavenumber than it does for bigger non-dimensional wavenumber. For a given structural parameter $a/\sqrt{Rh}$, the dynamic stress concentration factors decrease with the non–dimensional wavenumber α increasing in the interval of $\alpha \in [0{\cdot}1, 1{\cdot}0]$.

8. SUMMARY AND CONCLUSIONS

In the present work, based on the theory of the bending wave motion of open circular cylindrical shell, elastic wave scattering and dynamic stress concentrations in infinite open cylindrical shells with a circular cutout have been investigated by making use of small parameter perturbation methods and boundary element techniques. A boundary-integral equation method for solving this problem has been established. The boundary-integral equations and iterative imminent series of scattered waves around the cavity of a cylindrical shell have been given. The computational formula of dynamic stress concentration factors around the cutout has been developed. The main findings of this work are as follows:

(1) It is much effective to use small parameter perturbation methods and boundary–integral equation techniques to solve the problem of elastic wave scattering and dynamic stress concentrations on the contour of circular hole in cylindrical shells. Because the wave equations for cylindrical shell are more complex, one cannot use the method of wave functions expansion to solve this problem. But, with the method employed in this paper, one can finally get the approximately analytical solutions.
(2) For cylindrical shells with a circular cutout, it can be seen that the dynamic stress concentration factors are bigger than those of thin plates with a circular hole at a given non-dimensional wavenumber α.
(3) For a given non-dimensional wavenumber α, the dynamic stress concentration factor increases as the small perturbation parameter ε increases.
(4) Due to the influence of the membrane force, for the case of the small perturbation parameter ε being equal to zero, the dynamic stress concentration factors are larger than those of thin plates.
(5) The dynamic stress concentration factors increase as the structural parameter $a/\sqrt{Rh}$ increases for different non-dimensional wavenumbers.

ACKNOWLEDGMENTS

The authors would like to acknowledge the support of the National Science Foundation for this research work (Foundation No. 19972018).

REFERENCES

1. J. L. Sanders Jr. 1970 *American Society of Mechanical Engineers Journal of Applied Mechanics* **37**, 374–383. Cutouts in shallow shells.
2. B. Ghosh and J. N. Bandyopadhayay 1994 *Computers and Structures* **53**, 9–18. Bending analysis of conical shells with cutouts.
3. C. R. Steele, M. L. Steele 1983 *American Society of Mechanical Engineers Journal of Pressure Vessel Technology* **105**, 191–200. Stress analysis of nozzles in cylindrical vessels with external load.
4. C. R. Steele, M. L. Steele, A. Khathlan 1986 *American Society of Mechanical Engineers Journal of Pressure Vessel Technology* **108**, 436–442. An efficient computational approach for a large opening in a cylindrical vessel.

5. M. W. Hilburger, A. M. Waas, J. H. Starnes Jr. 1999 *American Institute of Aeronautics and Astronautics Journal* **37**, 232–237. Response of composite shells with cutouts to internal pressure and compression loads.
6. V. P. Dyke 1965 *American Institute of Aeronautics and Astronautics Journal* **3**. Stresses about a circular hole in a cylindrical shell.
7. S. T. Dennis and A. N. Palazotto 1990 *American Institute of Aeronautics and Astronautics Journal* **28**, 1082–1088. Static response of a cylindrical composite panel with cutouts using a geometrically nonlinear theory.
8. D. Y. Hwang, W. A. Foster Jr. 1992 *Journal of Sound and Vibration* **157**, 331–343. Analysis of axisymmetric free vibration of isotropic shallow spherical shells with a circular hole.
9. B. Sivasubramonian, A. M. Kulkarni and R. G. Venkateswara 1997 *Journal of Sound and Vibration* **200**, 227–234.
10. K. S. S. Ram and T. S. Babu 2001 *Composite Structures* **51**, 117–126. Study of bending laminated composite shells. Part II: shells with a cutout.
11. L. Flax and W. G. Neubauer 1978 *Journal of the Acoustical Society of America* **63**, 675–680. Reflection of elastic waves by a cylindrical cavity in an absorptive medium.
12. G. C. Gaunaurd and H. Überall 1978 *Journal of the Acoustical Society of America* **63**, 1699–1712. Theory of resonant scattering from spherical cavities in elastic and viscoelastic media.
13. P. Hillion 2001 *Journal of Sound and Vibration* **240**, 765–768. Scattering of an acoustic plane wave by a corrugated cylinder.
14. Y. Decanini, A. Folacci, P. Gabrielli and J. L. Rossi 1999 *Journal of Sound and Vibration* **221**, 785–804. Algebraic aspects of multiple scattering by two parallel cylinders: classification and physical interpretation of scattering resonances.
15. Y. H. Pao 1962 *American Society of Mechanical Engineers Journal of Applied Mechanics* **29**, 299–305. Dynamic stress concentrations in an elastic plate.
16. Y. H. Pao and C. C. Mow 1964 *American Institute of Aeronautics and Astronautics Journal* **2**, 2004–2010. Diffractions of flexural waves by a cavity in an elastic plate.
17. S. D. Bogan and M. K. Hinders 1993 *Journal of Composite Materials* **27**, 1272–1311. Dynamic stress concentrations in fiber-reinforced composites with interface layers.
18. D. K. Liu and C. Hu 1996 *Acta Mechanica Sinica* **12**, 169–185. Scattering of flexural wave and dynamic stress concentrations in Mindlin's thick plates.
19. P. Gabrielli and M. M. Finidori 2001 *Journal of Sound and Vibration* **241**, 423–439. Acoustic scattering by two spheres: multiple scattering and symmetry considerations.
20. H. Sato and Y. Shindo 2001 *International Journal of Solids and Structures* **38**, 2549–2571. Multiple scattering of plane elastic waves in a fiber-reinforced composite medium with graded interfacial layers.
21. K. Shankar 1998 *Journal of Sound and Vibration* **217**, 97–111. A study of the dynamic stress concentration factors of a flat plate for SEA applications.
22. C. C. Mow and L. J. Mente 1963 *American Society of Mechanical Engineers Journal of Applied Mechanics* **30**, 598–604. Dynamic stresses and displacements around cylindrical discontinuities due to plane harmonic shear waves.
23. C. C. Mow and J. W. Workman 1966 *American Society of Mechanical Engineers Journal of Applied Mechanics* **33**, 793–799. Dynamic stresses around a fluid-filled cavity.
24. D. K. Liu, B. Z. Gai and G. Y. Tao 1982 *Wave Motion* **4**, 293–304. Application of method of complex functions to dynamic stress concentrations.
25. V. J. Karam and J. C. F. Telles 1988 *Engineering Analysis Journal* **5**, 21–27. On boundary elements for Reissner's plate theory.
26. P. Lu and M. Huang 1992 *International Journal of Solids and Structures* **29**, 1273–1282. Boundary element analysis of shallow shells involving shear deformation.
27. Y. F. Rashed, M. H. Aliabadi and C. A. Brebbia 1998 *International Journal of Solids and Structures* **35**, 2229–2249. Hypersingular boundary element formulation for Reissner plates.
28. P. H. Wen, M. H. Aliabadi and A. Young 2000 *International Journal of Numerical Methods in Engineering* **48**, 1107–1125. Plane stress and plate bending coupling in BEM analysis of shallow shells.
29. T. Dirgantara and M. H. Aliabadi 1999 *International Journal of Numerical Methods in Engineering* **45**, 1257–1275. A new boundary element formulation for shear deformable shells analysis.
30. P. H. Wen, M. H. Aliabadi and A. Young 2000 *Engineering Analysis with Boundary Elements* **24**, 583–590. Application of dual reciprocity method to plates and shells.

31. T. Dirgantara and M. H. Aliabadi 2001 *International Journal of Solids and Structures* **38**, 7769–7800. Dual boundary element formulation for fracture mechanics analysis of shear deformable shells.
32. Y. H. Pao and C. C. Mow 1973 *Diffractions of Elastic Waves and Dynamic Stress Concentrations*. New York: Crane, Russak.
33. S. Bergman and M. Schiffer 1953 *Kernel Functions and Elliptic Differential Equations in Mathematical Physics*. New York: Academic Press.
34. C. A. Brebbia, J. C. F. Telles and L. C. Wrobel 1984 *Boundary Element Techniques—Theory and Applications in Engineering*. Berlin: Springer-Verlag.
35. C. A. Brebbia and S. Walker 1980 *Boundary Element Techniques in Engineering*. London: Butterworth & Co. (publishers) Ltd.